ON SOME FUNCTIONALS, II*

BY

S. SAKS

1. This article is primarily intended to correct the mistakes in §§4 and 5 of the former paper by the author *On some functionals*.† On the whole these errors do not affect the theorems themselves with the exception of the obviously false remarks that Theorem 4 and a part of Theorem 3 of S. F. hold for the space R of characteristic functions. In the present note we complete the gaps in the proofs of those theorems‡ and slightly strengthen Theorem 3 of S. F. in the part concerning the equal continuity of the operations considered (cf. below Theorem 3, (ii)). We also prove two theorems which were not stated in S. F., namely, Theorems 1 and 4 so as to obtain a more symmetric set of results.

2. We shall recall briefly the notation. Let $\{\xi_n(t)\}$ be a sequence of measurable functions on a measurable set U. Then

(i) $\{\xi_n(t)\}$ is *bounded in measure* on U if to every $\eta > 0$ there corresponds a number $M = M(\eta)$ such that

$$\text{meas } \underset{t}{E}[t\varepsilon U;\ |\xi_n(t)| > M] < \eta \qquad (n = 1, 2, \cdots);$$

(ii) $\{\xi_n(t)\}$ *converges in measure* on U if to every η there corresponds a $k = k(\eta)$ such that

$$\text{meas } \underset{t}{E}[t\varepsilon U;\ |\xi_n(t) - \xi_m(t)| > \eta] < \eta$$

whenever $n > k$, $m > k$;

(iii) $\{\xi_n(t)\}$ *has property* $B(\epsilon)$ on U, if there exists a number M independent of n, such that

$$\text{meas } \underset{t}{E}[t\varepsilon U;\ |\xi_n(t)| > M] < \epsilon \qquad (n = 1, 2, \cdots);$$

(iv) $\{\xi_n(t)\}$ *has property* $C(\epsilon)$ on U,§ if to every $\eta > 0$ there corresponds a $k = k(\eta)$ such that

* Presented to the Society, September 5, 1936; received by the editors April 27, 1936.

† These Transactions, vol. 35 (1932), pp. 549–556. This paper will be referred to as S. F.

‡ For the proof of Parts (i) and (ii) of Theorem 3 of S. F. see also the recent book by Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, Monografje Matematyczne, Warszawa-Lwow, 1935, pp. 24–26. The author's attention was called to these mistakes by Kaczmarz and by Gowurin. (Cf. Gowurin, *On sequences of indefinite integrals*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 930–936.)

§ We shall write $\{\xi_n(t)\}\varepsilon B(\epsilon)$ or $\varepsilon C(\epsilon)$ to indicate that the sequence $\{\xi_n(t)\}$ has property $B(\epsilon)$ or $C(\epsilon)$ respectively.

$$\text{meas}\, \underset{t}{E}[t\varepsilon U;\ |\xi_n(t) - \xi_m(t)| > \eta] < \epsilon, \qquad n > k,\ m > k.$$

In what follows we denote by I a measurable set of finite measure (e.g., an interval) and by E a Banach space. We shall consider sequences of functions $\{\xi_n(x, t)\}$ depending on $x\varepsilon E$ and $t\varepsilon I$. For each x fixed in E, $\xi_n(x, t)$ are finite and measurable functions of t in I. On the other hand, as functions of x, they are supposed to be linear operations on E; i.e., additive and continuous. This means that $\xi_n(x_1+x_2, t)=\xi_n(x_1, t)+\xi_n(x_2, t)$ almost everywhere on I, whenever $x_1\varepsilon E$, $x_2\varepsilon E$ and that $\xi_n(x, t)$ tends in measure to 0 on I when $|x|\to 0$ (for n fixed). In considering $\xi_n(x, t)$ as operations on E we shall often write $\xi_n(x, t)\equiv F_n(x)$.

The operations $F_n(x)=\xi_n(x, t)$ will be said to be *equally continuous* if to every $\eta>0$ there corresponds an $r>0$ such that

$$\text{meas}\, \underset{t}{E}[t\varepsilon I;\ |\xi_n(x, t)| > \eta] < \eta \qquad (n = 1, 2, \cdots),$$

whenever $|x| < r$. They will be said to be *equally continuous with respect to a set* $U\subset I$ if the condition above is satisfied with I replaced by U.

Finally $\{T_k\}$ $(k=1, 2, \cdots)$ will denote a sequence of measurable sets in I such that the characteristic functions of the sets T_k form an everywhere dense set in the space of all characteristic functions defined on I. Thus for each set Q and every $\eta>0$ there is a set T_k, $k=k(Q, \eta)$, such that meas $(Q-QT_k)<\eta$ and meas $(T_k-QT_k)<\eta$.

3. We have

LEMMA 1. *Let* $\sum_1^\infty \epsilon_k$ *be a converging series of positive numbers,* $\{P_q\}$ *a sequence of measurable sets in I and*

$$P = \limsup_k P_k = \prod_{k=1}^{\infty} \sum_{q=k}^{\infty} P_q. \tag{3.1}$$

Then, if a sequence $\{\xi_n(t)\}\varepsilon B(\epsilon_k)$ *on* P_k $(k=1, 2, \cdots)$, *it is bounded in measure on* P; *and if a sequence* $\{\xi_n(t)\}\varepsilon C(\epsilon_k)$ *on* P_k $(k=1, 2, \cdots)$, *it converges in measure on* P.

Let

$$Q_k = \sum_{q=k}^{\infty} P_q, \qquad Q_{k,s} = \sum_{q=k}^{s} P_q,$$

and let η be an arbitrary positive number. Choose k_0 and next s_0, so that

$$\sum_{i=k_0}^{\infty} \epsilon_i < \frac{\eta}{2}, \qquad \text{meas}\,(Q_{k_0} - Q_{k_0,s_0}) < \frac{\eta}{2}. \tag{3.2}$$

Suppose now that the sequence under consideration $\varepsilon B(\epsilon_k)$ on P_k. Then there is a number M such that

$$\operatorname{meas} \underset{t}{E}[t\varepsilon Q_{k_0,s_0};\ |\ \xi_n(t)\ | > M] < \sum_{q=k_0}^{s_0} \epsilon_q < \frac{\eta}{2} \qquad (n = 1, 2, \cdots);$$

thus, by (3.1) and (3.2)

$$\operatorname{meas} \underset{t}{E}[t\varepsilon P;\ |\ \xi_n(t)\ | > M] \leqq \operatorname{meas} \underset{t}{E}[t\varepsilon Q_{k_0};\ \ \xi_n(t)\ | > M] < \eta \qquad (n = 1, 2, \cdots);$$

i.e., the sequence $\{\xi_n(t)\}$ is bounded in measure on P.

Next suppose that the sequence $\{\xi_n(t)\}\varepsilon C(\epsilon_k)$ on P_k. Then there is a positive integer n_0 such that

$$\operatorname{meas} \underset{t}{E}[t\varepsilon Q_{k_0,s_0};\ |\ \xi_n(t) - \xi_m(t)\ | > \eta] < \sum_{i=k_0}^{s_0} \epsilon_i < \frac{\eta}{2}, \qquad n \geqq n_0, \quad m \geqq n_0,$$

and so, by (3.1) and (3.2)

$$\operatorname{meas} \underset{t}{E}[t\varepsilon P;\ |\ \xi_n(t) - \xi_m(t)\ | > \eta] \leqq \operatorname{meas} \underset{t}{E}[t\varepsilon Q_{k_0};\ |\ \xi_n(t) - \xi_m(t)\ | > \eta] < \eta, \qquad n \geqq n_0,\ m \geqq n_0.$$

LEMMA 2. *If a sequence $\{\xi_n(x, t)\}\varepsilon B(\epsilon)$ on a measurable set $P \subset I$ for every x belonging to a set H of the second category in E, then, to every $\eta > 0$, there corresponds an $r > 0$ such that for an arbitrary $x\varepsilon E$ with $|x| < r$,*

$$\operatorname{meas} \underset{t}{E}[t\varepsilon P;\ |\ \xi_n(x, t)\ | > \eta] < 3\epsilon \qquad (n = 1, 2, \cdots). \tag{3.3}$$

Let H_m denote the set of all $x\varepsilon H$ such that

$$\operatorname{meas} \underset{t}{E}[t\varepsilon P;\ |\ \xi_n(x, t)\ | > m] < \epsilon \qquad (n = 1, 2, \cdots).$$

We have $H = \sum_1^\infty H_m$, and so there is a positive integer m_0 such that H_{m_0} is of the second category. Since $\xi_n(x, t)$ is continuous in x on E, the inequalities

$$\operatorname{meas} \underset{t}{E}[t\varepsilon P;\ |\ \xi_n(x, t)\ | > m_0] \leqq \epsilon \qquad (n = 1, 2, \cdots)$$

hold for all $x\varepsilon \overline{H}_{m_0}$, and consequently for all x in a sphere $K_0 \subset \overline{H}_{m_0}$. Let r_0 be the radius of K_0. By the linearity of the operations $\xi_n(x, t)$, we have

$$\operatorname{meas} \underset{t}{E}\ [t\varepsilon P;\ |\ \xi_n(x, t)\ | > 2m_0] \leqq 2\epsilon \qquad (n = 1, 2, \cdots),$$

whenever $|x| < r_0$, and therefore condition (3.3) is satisfied with $r = \eta r_0/(2m_0)$.

4. We now prove the following theorem:

THEOREM 1. *For any given sequence $\{\xi_n(x, t)\}$ there exists a set A in I such that*

(i) *the sequence $\{\xi_n(x, t)\}$ is bounded in measure on A for all $x\varepsilon E$;*

(ii) *the operations* $F_n(x)=\xi_n(x, t)$ *are equally continuous with respect to* A;

(iii) *for every* x *in* E, *except perhaps for a set of the first category in* E, *the sequence* $\{\xi_n(x, t)\}$ *is bounded in measure on no set in* $I-A$ *of positive measure.*

Denote by α_0 the least upper bound of all numbers α with the property that there exists a set $H(\alpha)$ of the second category in E such that for each x in $H(\alpha)$ the sequence $\{\xi_n(x, t)\}$ is bounded in measure on a set of measure $>\alpha$. Let q be a positive integer. We shall prove first that there exists a set H_q of the second category in E and a set P_q in I of measure $>\alpha_0-1/q^2$ such that

$$\{\xi_n(x, t)\}\,\varepsilon B(1/q^2) \text{ on } P_q, \quad x\varepsilon H_q. \tag{4.1}$$

Indeed to every $x\varepsilon H(\alpha_0-1/q^2)$ we can attach a set T_k, $k=k(x)$ (see §2), such that

$$\{\xi_n(x, t)\}\,\varepsilon B(1/q^2) \text{ on } T_k, \quad \text{meas } T_k > \alpha_0 - \frac{1}{q^2}. \tag{4.2}$$

Let $H_{q,k}$ denote the set of all $x\varepsilon H(\alpha_0-1/q^2)$ for which (4.2) is satisfied for a fixed k. Since $H(\alpha_0-1/q^2)=\sum_{k=1}^{\infty}H_{q,k}$ is of the second category, there exists a $k=k_q$ such that H_{q,k_q} is of the second category. On putting $H_q=H_{q,k_q}$ and $P_q=T_{k_q}$, we see at once that H_q and P_q satisfy condition (4.1).

We shall prove next that for every x in E

$$\{\xi_n(x, t)\}\,\varepsilon B(8/q^2) \text{ on } P_q. \tag{4.3}$$

To show this consider an arbitrary point $x_0\varepsilon \overline{H}_q$. By (4.1) and Lemma 2

$$\text{meas } \underset{t}{E}[t\varepsilon P_q;\ |\xi_n(x_0 - x_1, t)| > 1] < \frac{3}{q^2} \qquad (n = 1, 2, \cdots), \tag{4.4}$$

if $|x_0-x_1|$ is sufficiently small. Hence we can choose $x_1\varepsilon H_q$ so as to satisfy (4.4). Since $\{\xi_n(x_1, t)\}\,\varepsilon B(1/q^2)$ on P_q this implies that $\{\xi_n(x_0, t)\}\,\varepsilon B(4/q^2)$ on P_q. But x_0 is an arbitrary element of the set $\overline{H}_q$ which certainly contains interior points (a sphere), whence it follows that $\{\xi_n(x, t)\}\,\varepsilon B(8/q^2)$ on P_q for all x in E.

Now set $A=\prod_{k=1}^{\infty}\sum_{q=k}^{\infty}P_q$. It results from (4.3) and Lemma 1 that the sequence $\{\xi_n(x, t)\}$ is bounded in measure on A, i.e., that A satisfies condition (i) of the theorem.

Next let η be an arbitrary positive number, and let, as in the proof of Lemma 1,

$$Q_k = \sum_{q=k}^{\infty} P_q, \qquad Q_{k,s} = \sum_{q=k}^{s} P_q. \tag{4.5}$$

Let k_0 and s_0 be positive integers such that

$$(4.6) \qquad 24/(k_0 - 1) < \frac{\eta}{2}, \quad \text{meas}\,(Q_{k_0} - Q_{k_0,s_0}) < \frac{\eta}{2}.$$

By (4.3), Lemma 2 and (4.6), there exists an $r=r(\eta)$ such that, for $|x|<r$,

$$\text{meas}\,\underset{t}{E}[t\varepsilon Q_{k_0,s_0};\ |\,\xi_n(x,t)\,| > \eta] < 24\sum_{q=k_0}^{s_0} q^{-2} < \frac{\eta}{2} \qquad (n = 1, 2, \cdots),$$

and thus, again by (4.6)

$$\text{meas}\,\underset{t}{E}[t\varepsilon A;\ |\,\xi_n(x,t)\,| > \eta] \leqq \text{meas}\,\underset{t}{E}[t\varepsilon Q_{k_0};\ |\,\xi_n(x,t)\,| > \eta] < \eta$$

which is condition (ii).

Finally, in order to prove condition (iii) suppose that there is a set H of the second category in E such that for every $x\varepsilon H$ the sequence $\{\xi_n(x,t)\}$ is bounded in measure on a set $Q(x) \subset I-A$ of positive measure. Then, for every $x\varepsilon H$ the sequence $\{\xi_n(x,t)\}$ would be bounded in measure on $A+Q(x)$. Now since

$$\text{meas}\,[A + Q(x)] > \text{meas}\,A \geqq \limsup_q\ \text{meas}\,P_q \geqq \lim_q\,(\alpha_0 - 1/q^2) = \alpha_0,$$

we have

$$H = \sum_{n=1}^{\infty} H_n, \quad H_n = \underset{x}{E}[x\varepsilon H; \text{meas}\,\{A + Q(x)\} \geqq \alpha_0 + 1/n].$$

Since at least one of the sets H_n is of the second category, this contradicts the definition of α_0.

5. Next we prove

THEOREM 2. *There exists a set B in I such that*

(i) *for all x in E, the sequence $\{\xi_n(x,t)\}$ converges in measure on B;*

(ii) *for every x in E, except perhaps for a set of the first category in E, the sequence $\{\xi_n(x,t)\}$ converges in measure on no set in $I-B$ of positive measure.*

First observe that for every $x\varepsilon E$, except perhaps for a set of the first category, the sequence $\{\xi_n(x,t)\}$ does not converge in measure on any set of positive measure contained in $I-A$ (where A is the set defined in Theorem 1). Hence to prove Theorem 2 we may assume that $I=A$, or, which amounts to the same (see Theorem 1, (ii)) that the operations $F_n(x)=\xi_n(x,t)$ are equally continuous with respect to the whole set I.

From now on we shall follow the line of the proof of Theorem 1. Denote by β_0 the least upper bound of all numbers β with the property that there exists a set $H(\beta)$ of the second category in E such that for every $x\varepsilon H(\beta)$ the sequence $\{\xi_n(x,t)\}$ converges in measure on a set of measure $>\beta$. Let q be an arbitrary positive integer, and let $H_{q,k}$ denote the set of all $x \varepsilon H(\beta_0 - 1/q^2)$ such that $\{\xi_n(x, t)\} \varepsilon C(1/q^2)$ on a set T_k (see §2) of measure $> \beta_0 - 1/q^2$. Clearly $H(\beta_0 - 1/q^2) = \sum_{k=1}^{\infty} H_{q,k}$, and thus there is a $k = k_q$ such that H_{q,k_q} is of the second category. Hence upon putting $H_q = H_{q,k_q}$ and $P_q = T_{k_q}$ we see that there exists a set H_q of the second category in E and a set P_q in I of measure $> \beta_0 - 1/q^2$ such that

$$\{\xi_n(x, t)\} \varepsilon C(1/q^2) \text{ on } P_q, \quad x \varepsilon H_q. \tag{5.1}$$

Now as in the proof of Theorem 1 we shall show that for every x in E

$$\{\xi_n(x, t)\} \varepsilon C(6/q^2) \text{ on } P_q. \tag{5.2}$$

Indeed, let $x_0 \varepsilon \overline{H}_q$ and η be an arbitrary positive number. The equal continuity of the operations $F_n(x)$ implies the existence of an $x_1 \varepsilon H_q$ sufficiently near to x_0 such that

$$\text{meas} \underset{t}{E}\left[t \varepsilon I;\ |\xi_n(x_1 - x_0, t)| > \frac{\eta}{6}\right] < \frac{1}{q^2} \qquad (n = 1, 2, \cdots). \tag{5.3}$$

Again, in view of (5.1) there exists a positive integer n_0 such that, for $n \geqq n_0$, $m \geqq n_0$,

$$\text{meas} \underset{t}{E}\left[t \varepsilon P_q;\ |\xi_m(x_1, t) - \xi_n(x_1, t)| > \frac{\eta}{6}\right] < \frac{1}{q^2}.$$

Thus, by (5.3),

$$\text{meas} \underset{t}{E}\left[t \varepsilon P_q;\ |\xi_m(x_0, t) - \xi_n(x_0, t)| > \frac{\eta}{2}\right] < \frac{3}{q^2}, \qquad m \geqq n_0,\ n \geqq n_0. \tag{5.4}$$

Let K_0 be an arbitrary sphere contained in $\overline{H}_q$ and r_0 the radius of K_0. Since (5.4) holds for all elements $x_0 \varepsilon K_0$, we have, by the linearity of the transformations $F_n(x)$,

$$\text{meas} \underset{t}{E}[t \varepsilon P_q;\ |\xi_n(x, t) - \xi_m(x, t)| > \eta] < \frac{6}{q^2}, \quad m \geqq n_0,\ n \geqq n_0,\ |x| < r_0.$$

This means however that $\{\xi_n(x, t)\} \varepsilon C(6/q^2)$ on P_q whenever $|x| < r_0$ and consequently for every x in E.

Now let

$$B = \prod_{k=1}^{\infty} \sum_{q=k}^{\infty} P_q.$$

By Lemma 1 the sequence $\{\xi_n(x, t)\}$ converges in measure on B for every x. Hence, the set B satisfies condition (i). To establish condition (ii) suppose that there is a set H of the second category such that for every $x\varepsilon H$ the sequence $\{\xi_n(x, t)\}$ converges in measure on a set $Q(x) \subset I-B$ of positive measure. Then, for every $x\varepsilon H$ the sequence would converge in measure on $B+Q(x)$. But

$$\text{meas } [B + Q(x)] > \text{meas } B \geqq \limsup_q \text{ meas } P_q \geqq \beta_0,$$

which contradicts the definition of the number β_0 (cf. the proof of Theorem 1). Condition (ii) is thus established.

6. We now have

THEOREM 3. *There exists a set C in I such that*

(i) *for all x in E, $\sup_n|\xi_n(x, t)| < \infty$ almost everywhere in C;*

(ii) *to every $\eta > 0$ there corresponds an $r > 0$ such that*

$$\text{meas } \underset{t}{E}[t\varepsilon C; \sup_n |\xi_n(x, t)| > \eta] < \eta, \qquad |x| < r;$$

(iii) *for every x in E, except perhaps for a set of the first category in E,*

$$\sup_n |\xi_n(x, t)| = \infty$$

almost everywhere in $I-C$.

Let γ_0 be the least upper bound of all numbers γ such that the set $H(\gamma)$ of x for which

$$\text{meas } \underset{t}{E}[\sup_n |\xi_n(x, t)| < \infty] > \gamma$$

is of the second category. Let q be an arbitrary positive integer and let $H_{q,p,k}$ denote the set of all $x\varepsilon H(\gamma_0 - 1/q^2)$ such that

$$\text{meas } \underset{t}{E}[t\varepsilon T_k; \sup_n |\xi_n(x, t)| > p] < \frac{1}{q^2}, \qquad \text{meas } T_k > \gamma_0 - \frac{1}{q^2}.$$

It is clear that $H(\gamma_0 - 1/q^2) = \sum_{p,k=1}^{\infty} H_{q,p,k}$ and so there exist integers $k = k_q$ and $p = p_q$ such that H_{q,p_q,k_q} is of the second category. Put $H_q = H_{q,p_q,k_q}$ and $P_q = T_{k_q}$. Thus for every $x\varepsilon H_q$

$$\text{meas } \underset{t}{E}[t\varepsilon P_q; \sup_n |\xi_n(x, t)| > p_q] < \frac{1}{q^2}$$

while meas $P_q > \gamma_0 - 1/q^2$. Hence, by continuity of $\xi_n(x, t)$, for every $x\varepsilon \overline{H}_q$ and all s,

$$\text{meas } \underset{t}{E}[t\varepsilon P_q; \sup_{n\leqq s} |\xi_n(x, t)| > p_q] \leqq \frac{1}{q^2},$$

and therefore, for all $x \varepsilon \overline{H}_q$,

$$\text{(6.1)} \qquad \operatorname{meas} \underset{t}{E}[t \varepsilon P_q; \sup_n |\xi_n(x, t)| > p_q] \leqq \frac{1}{q^2}.$$

Let K_0 be a sphere contained in $\overline{H}_q$. If r_0 is the radius of K_0, (6.1) implies in view of the linearity of $\xi_n(x, t)$,

$$\operatorname{meas} \underset{t}{E}[t \varepsilon P_q; \sup_n |\xi_n(x, t)| > 2p_q] \leqq \frac{2}{q^2},$$

no matter what $x \varepsilon E$, provided $|x| < r_0$, and consequently, for any $\sigma > 0$,

$$\text{(6.2)} \qquad \operatorname{meas} \underset{t}{E}[t \varepsilon P_q; \sup_n |\xi_n(x, t)| > \sigma] \leqq \frac{2}{q^2}, \qquad |x| < \frac{r_0 \sigma}{2p_q}.$$

Now, again by linearity of $\xi_n(x, t)$ it results from (6.2) that for every $x \varepsilon E$, $\sup_n |\xi_n(x, t)| < \infty$ for all $t \varepsilon P_q$, with the exception of at most a subset of measure $\leqq 2/q^2$. Hence, upon putting $C = \prod_{k=1}^{\infty} \sum_{q=k}^{\infty} P_q$ we see at once that, for every $x \varepsilon E$, $\sup_n |\xi_n(x, t)| < \infty$ almost everywhere in C. Thus C satisfies condition (i) of the theorem.

Next, as in the proof of Lemma 1, let

$$Q_k = \sum_{q=k}^{\infty} P_q, \qquad Q_{k,s} = \sum_{q=k}^{s} P_q.$$

Let η be an arbitrary positive number, and let k_0 and s_0 be positive integers such that

$$\text{(6.3)} \qquad \frac{2}{k_0 - 1} < \frac{\eta}{2}, \ \operatorname{meas} (Q_{k_0} - Q_{k_0, s_0}) < \frac{\eta}{2}.$$

In virtue of (6.2) there is an $r > 0$ such that whenever $|x| < r$,

$$\operatorname{meas} \underset{t}{E}[t \varepsilon Q_{k_0, s_0}; \sup_n |\xi_n(x, t)| > \eta] \leqq \sum_{q=k_0}^{s_0} 2q^{-2} < \frac{\eta}{2}$$

and therefore, by (6.3),

$$\operatorname{meas} \underset{t}{E}[t \varepsilon C; \sup_n |\xi_n(x, t)| > \eta]$$

$$\leqq \operatorname{meas} \underset{t}{E}[t \varepsilon Q_{k_0}; \sup_n |\xi_n(x, t)| > \eta] < \eta, \qquad |x| < r.$$

Thus condition (ii) for C is established. Finally condition (iii) follows at once from the definition of γ_0, since

$$\operatorname{meas} C \geqq \limsup_q \operatorname{meas} P_q \geqq \gamma_0.$$

7. Next we have

THEOREM 4. *There exists a set D in I such that*

(i) *for all x in E the sequence* $\{\xi_n(x, t)\}$ *converges almost everywhere on D*;

(ii) *for every x in E except perhaps for a set of the first category, the sequence* $\{\xi_n(x, t)\}$ *diverges almost everywhere on* $I-D$.

First observe that for every x, except perhaps for a set of the first category, the sequence $\{\xi_n(x, t)\}$ diverges almost everywhere on $I-C$, where C is the set defined in Theorem 3. Thus, without loss of generality, we may assume that $I=C$.

For a fixed $x\varepsilon E$ let $\Gamma(x)$ denote the subset of I on which $\{\xi_n(x, t)\}$ converges. Further let $H(\delta)$ denote the set of all $x\varepsilon E$ such that meas $\Gamma(x)>\delta$, $\delta>0$, and let δ_0 be the upper bound of the numbers δ for which $H(\delta)$ is of the second category.

Now, let q be an arbitrary positive integer and $H_{q,k}$ the set of all $x\varepsilon H(\delta_0-1/q^2)$ such that

$$\text{meas } T_k > \delta_0 - \frac{1}{q^2}, \quad \text{meas } (T_k - T_k\Gamma(x)) < \frac{1}{q^2}.$$

We thus have $H(\delta_0-1/q^2)=\sum_{k=1}^{\infty}H_{q,k}$, and so there exists a $k=k_q$ such that $H_{q,k}$ is of the second category. Put $H_q=H_{q,k_q}$ and $P_q=T_{k_q}$. Then

$$\text{meas } (P_q - P_q\Gamma(x)) \leqq \frac{1}{q^2}, \qquad x\varepsilon H_q, \tag{7.1}$$

but we shall show that the latter inequality holds for every $x\varepsilon \overline{H}_q$. Indeed, let $x_0\varepsilon\overline{H}_q$ and η be an arbitrary positive number. In virtue of Theorem 3 (condition (ii) with $C=I$) we can find an $x_1\varepsilon H_q$ such that

$$\text{meas } \underset{t}{E}[t\varepsilon I;\ \sup_n |\xi_n(x_0 - x_1, t)| > \eta] < \eta.$$

Again, by (7.1) there is a positive integer $n_0=n_0(\eta)$ such that

$$\text{meas } \underset{t}{E}[t\varepsilon P_q;\ \sup_{n\geqq n_0} |\xi_n(x_1, t) - \xi_{n_0}(x_1, t)| > \eta] \leqq \frac{1}{q^2} + \eta,$$

whence

$$\text{meas } \underset{t}{E}[t\varepsilon P_q;\ \sup_{n\geqq n} |\xi_n(x_0, t) - \xi_{n_0}(x_0, t)| > 3\eta] \leqq \frac{1}{q^2} + 3\eta.$$

Since $\eta>0$ is arbitrary this implies the convergence of the sequence $\{\xi_n(x, t)\}$ for every $x\varepsilon\overline{H}_q$ and for all $t\varepsilon P_q$, with the exception of at most a subset of measure $\leqq 1/q^2$. Since $\overline{H}_q$ certainly contains a sphere the same holds for every $x\varepsilon E$ by the linearity of $\xi_n(x, t)$. Therefore, for each $x\varepsilon E$ the sequence converges almost everywhere on the set $D=\prod_{k=1}^{\infty}\sum_{q=k}^{\infty}P_q$. Thus the set D satisfies condition (i) of the theorem and condition (ii) follows at once since

$$\text{meas } D \geqq \limsup_q \text{meas } P_q \geqq \delta_0.$$

8. In view of Theorems 1, 2, 3, and 4 to every sequence $\{\xi_n(x, t)\}$ of the type considered there are attached four sets A, B, C, and D. We obviously have $B \subset A$ and $D \subset C$ (except for sets of measure zero). However, on account of condition (ii) of Theorem 1 (or 3) we have $B=A$ (or $D=C$) whenever E contains an everywhere dense set E_1 such that the sequence $\{\xi_n(x, t)\}$ converges in measure (or converges almost everywhere) in I for every x belonging* to E_1. It follows that if E is a separable space then from the sequence $\{\xi_n(x, t)\}$ a subsequence $\{\xi_n'(x, t)\}$ may be extracted which converges almost everywhere in BC for every $x\varepsilon E$; in fact it is sufficient to select the required subsequence so that it converges almost everywhere in B for every x belonging to an everywhere dense, denumerable set in E.

9. The theorems of the preceding sections do not hold† in general, if the linear space E is replaced by the space R of characteristic functions. However, for the latter we have the following theorem:

THEOREM 5. (i) *If for all x belonging to a set H of the second category in R the sequence $\{F_n(x)=\xi_n(x, t)\}$, $x\varepsilon R$, $t\varepsilon I$, converges in measure on I, then the operations $F_n(x)$ are equally continuous in I, i.e., for every $\eta>0$ there exists an $r>0$ such that*

$$\text{meas } \underset{t}{E}[t\varepsilon I;\ |\xi_n(x, t)| > \eta] \leqq \eta \qquad (n = 1, 2, \cdots),$$

whenever $|x| < r$.

(ii) *If for all x belonging to a set H of the second category in R, the sequence $\{\xi_n(x, t)\}$ converges almost everywhere on I, then for every $\eta>0$ there exists an $r>0$ such that*

$$\text{meas } \underset{t}{E}[t\varepsilon I;\ \sup_n |\xi_n(x, t)| > \eta] < \eta,$$

whenever $|x| < r$.

We shall confine ourselves to the proof of statement (ii) of the theorem. The proof of statement (i) is the same as that of the lemma of S. F., p. 555.

* Cf. Banach, *Sur la convergence presque partout de fonctionnelles linéaires*, Bulletin des Sciences Mathématiques, vol. 50 (1926), pp. 27–32, 36–43; Saks, *Sur les fonctionnelles de M. Banach et leur application aux développements des fonctions*, Fundamenta Mathematicae, vol. 10 (1927), pp. 186–196; Mazur and Orlicz, *Über Folgen linearer Operationen*, Studia Mathematica, vol. 4 (1933), pp. 152–157; esp. p. 157; Kaczmarz and Steinhaus, op. cit., pp. 177–178.

† See Gowurin, loc. cit.

Let η be any positive number and k a positive integer; let H_k denote the set of all $x \varepsilon H$ such that

$$\text{meas } \underset{t}{E}\left[t \varepsilon I;\ \sup_{n \geqq k} \mid \xi_n(x, t) - \xi_k(x, t) \mid > \frac{\eta}{4}\right] \leqq \frac{\eta}{4}.$$

We have $H = \sum_1^\infty H_k$ and thus there exists a $k = k_0$ such that H_{k_0} is of the second category. Hence

$$\text{meas } \underset{t}{E}\left[t \varepsilon I;\ \sup_{n \geqq k_0} \mid \xi_n(x, t) - \xi_{k_0}(x, t) \mid > \frac{\eta}{4}\right] \leqq \frac{\eta}{4},$$

for each $x \varepsilon H_{k_0}$, and therefore, by continuity (cf. the proof of Theorem 3), for all $x \varepsilon \overline{H}_{k_0}$. Let K_0 be a sphere which is contained in $\overline{H}_{k_0}$ and let r_0 be its radius. For every element $x \varepsilon R$, $|x| < r_0$, there are two elements x_1 and x_2 in R such* that $x_1 \varepsilon K_0$, $x_2 \varepsilon K_0$ and $x_1 = x_2 + x$. It is readily seen that, no matter what x is, $|x| < r_0$,

$$\text{(9.1)} \qquad \text{meas } \underset{t}{E}\left[t \varepsilon I;\ \sup_{n \geqq k_0} \mid \xi_n(x, t) - \xi_{k_0}(x, t) \mid > \frac{\eta}{2}\right] \leqq \frac{\eta}{2}.$$

Let now $r < r_0$ be a positive number such that

$$\text{meas } \underset{t}{E}\left[t \varepsilon I;\ \mid \xi_n(x, t) \mid > \frac{\eta}{2}\right] \leqq \frac{\eta}{2k_0} \qquad (n = 1, 2, \cdots, k_0),$$

whenever $|x| < r$. Then in view of (9.1), for every x, $|x| < r$,

$$\text{meas } E_t[t \varepsilon I;\ \sup_n \mid \xi_n(x, t) \mid > \eta] \leqq \eta.$$

* We put $x_1 = x + x_0(1 - x)$, $x_2 = x_0(1 - x)$ where x_0 is the center of the sphere K_0.

The University of Warsaw,
Warsaw, Poland